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Estimating damage laws from bend-test data

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Bend-test curves were calculated for materials obeying three very different damage laws. One result of the calculations is that the neutral axis migrates towards the compression face as the load-point displacement increases. For the three damage laws investigated, there is a simple relationship between the location of the neutral axis and the secant modulus derived from the bend-test record; this relationship is quite insensitive to the exact form of the damage law. The stress at a point 20 per cent of the beam depth from the tensile outer fibre is also fairly insensitive to the form of the damage law. Combining these two observations, it is possible to use bend-test data to make a good estimate of the tensile stress–strain curve of a material subject to damage.

1. Introduction

Damage theory has been developed to describe the difference in tensile and compressive behaviour of materials. For example, otherwise identical test specimens containing arrays of microcracks will be more compliant in tension than in compression. More generally, damage theory is intended to account for such processes as the formation of pores and microcracks in ceramics deforming by creep under tensile load [1]. The theory is particularly important in analysing bending where both stress states are developed in a single specimen. As examples of such applications Krajcinovic [2] has successfully compared tensile and bending strengths of concrete, and Talty and Dirks [3] have explained the difference in creep rates of Si_3N_4 in compression and bending.

A problem with application of damage theory to bending is that the form of the damage relationship must be assumed. This paper describes a method for approximating the relationship based solely on bend data. The method utilizes calculations which indicate that certain characteristics of the experimental bend-test record do not depend on the form of the constitutive relationship.

2. Analysis

Constitutive relationships associated with three different damage laws were chosen for use in the analysis. The criterion for selection was to provide a wide variation in the moment/curvature relation. The damage laws are:

1. An adaptation of Finnie's [4] creep-damage law to rate-independent deformation:

$$\sigma = AE\epsilon \begin{cases} A = 1, & \text{compression} \\ A < 1, & \text{tension} \end{cases} \quad (1)$$

Equation 1 will be referred to as the "elastic" relationship. For example, it describes the situation where the material contains pre-existing cracks which open, but do not grow or multiply under tensile loads.

2. The "elastic–plastic" relationship of Hoagland, *et al.* [5]

$$\begin{aligned} \epsilon &= \sigma/E; & \epsilon &\leqslant S/E \\ \sigma &= S; & \epsilon &\geqslant S/E \end{aligned} \quad (2)$$

Equation 2 describes a possible case of flaw formation and growth initiating at some fixed tensile stress, S.

3. The conventional linear-damage law, as used by Krajcinovic [2, 6], for example, corresponds to the case where damage is proportional to the tensile strain and is characterized by a stress, D:

$$\begin{aligned} \sigma &= E\epsilon; & \epsilon &\leqslant 0 \\ \sigma &= E\epsilon(1 - E\epsilon/D); & \epsilon &\geqslant 0 \end{aligned} \quad (3)$$

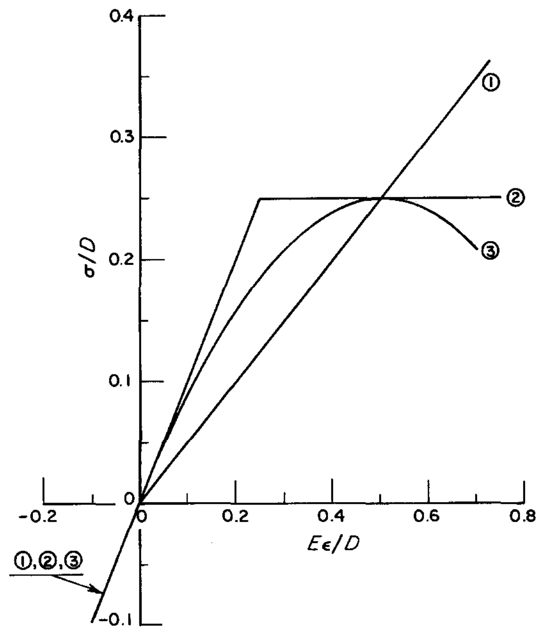

Figure 1 Constitutive relationships for three different damage laws.

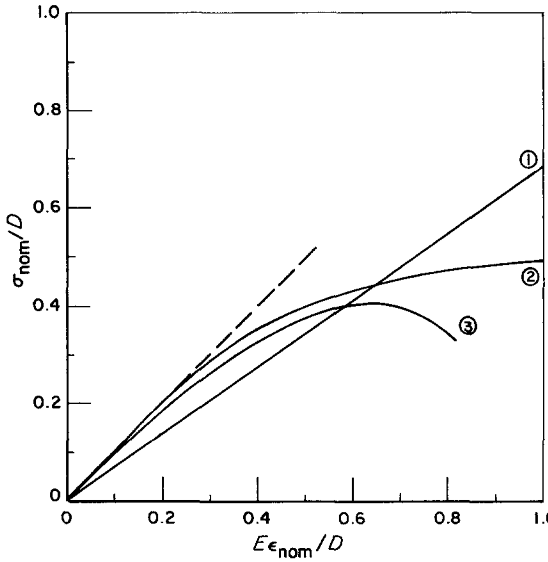

Figure 2 Moment–curvature curves for three different damage laws. Dashed line is damage-free curve.

Note that σ and ϵ in Equations 1 to 3 are engineering stress and strain, respectively.

Fig. 1 shows the form of the three different constitutive relationships. For comparison purposes the constants in Equations 1 and 2 have been chosen so that all curves pass through the maximum of Equation 3. To do so, A (Equation 1) was set equal to 1/2 and S (Equation 2) was set equal to $D/4$. Moment–curvature relationships were derived from standard beam theory using a procedure similar to that of Shetty and Gordon [7] for damage-free creep specimens. The procedure is described in the Appendix. Briefly, the calculation involves the evaluation of stress and strain distributions in terms of the location of the neutral axis. This procedure follows the observation of Krajcinovic [2] who noted that the location of the neutral axis moves away from the tension face as loading proceeds*. The separation of the neutral axis from the tension face, y_T, was expressed in terms of a parameter, β, where

$$y_T = \frac{H}{1+\beta} \tag{4}$$

In practice, load–displacement records can be converted into moment–curvature relationships which in turn can be expressed in terms of β and related to nominal stress (σ_{nom}) and strain (ϵ_{nom}), defined as

$$\sigma_{nom} = \frac{6M}{BH^2}, \tag{5}$$

where M is applied moment, B is beam thickness, and H is beam depth; and

$$\epsilon_{nom} = \frac{H}{2\rho} \tag{6}$$

where ρ is the radius of curvature. The resulting nominal stress/nominal strain relationships are given in Table I. The result for the "elastic" law has a fundamental difference from those for the other laws: in the elastic case the neutral axis is displaced from the beam mid-height but does not move as loading proceeds.

Fig. 2 does show the desired difference in moment–curvature relationships, varying from linear for the "elastic" law of Equation 1, approaching an asymptote ($\sigma_{nom} = 3D/4$) for the "elastic–plastic" law of Equation 2, to passing through a maximum for the linear-damage law of Equation 3. The calculation for the linear-damage law was terminated when the outer-fibre stress dropped to zero, corresponding to $\epsilon_{nom} = 0.788D/E$.

*Krajcinovic [2] also stated that the motion of the neutral axis eventually changes direction. This result arose from carrying the linear damage calculations beyond the point where the outer-fibre stress passes through zero ($\epsilon = D/E$ in Equation 1) creating spurious negative outer-fibre stresses.

TABLE I Characteristic relationships for bend-bars obeying different damage laws

	Elastic law, Equation 1	Elastic–plastic law, Equation 2*	Linear damage law, Equation 3
Nominal stress against nominal strain $\sigma_{nom}/E\epsilon_{nom}$	$\left(\frac{2\beta}{1+\beta}\right)^2$	$\frac{4\alpha^2(3-\alpha)}{(1+\alpha)^3}$	$\frac{(4)(\beta^2+\beta/8-1/8)}{(1+\beta)^2}$
Stress at any depth, z, below the tensile outer fibre, $\sigma(z)/\sigma_{nom}$	$\frac{(1+\beta)}{2}\left[1-\frac{(1+\beta)z}{H}\right]$	$1-\frac{2z}{H}$; $\alpha=1$ $\left(\frac{1+\alpha}{3-\alpha}\right)\left(\frac{1+\alpha^2}{2\alpha^2}\right) \times \left[1-\frac{z}{H}\frac{(1+\alpha)^2}{1+\alpha^2}\right]$; $z/H \leqslant (1-\alpha)/(1+\alpha)$ $\frac{1+\alpha}{3-\alpha}$; $z/H \geqslant \frac{1-\alpha}{1+\alpha}$	$\frac{(f)(1-f)}{(3)(1-\beta)(\beta^2+\beta/8-1/8)}$, where $f=\frac{3}{2}(1-\beta^2)\left[1-\frac{z}{H}(1+\beta)\right]$

*α is the distance from the neutral axis to the point where the "yield stress" is reached; $\beta = 2\alpha/(1+\alpha^2)$.

From these calculations it is clear that the moment–curvature relationship can be calculated if one knows the damage law in advance. Provided the elastic modulus is known, the damage law can be evaluated from a tensile test. However, tensile testing of brittle materials requires great care whereas bend testing is considerably simpler. For this reason, it is useful to ask whether the damage law can be deduced if the only information available is the bend-test result. In other words, given curves such as shown in Fig. 2, how can curves such as those shown in Fig. 1 be generated?

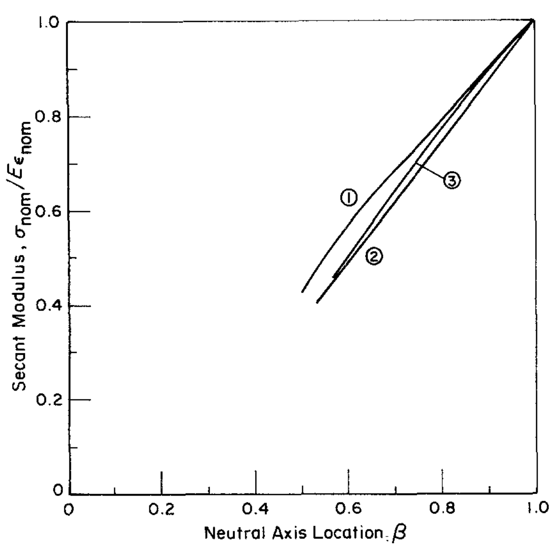

Figure 3 Distance of neutral axis from tension surface is given by $H/(1+\beta)$.

Consider first the calculation of strain. A complete description of the strain distribution is possible, provided the curvature and the location of the neutral axis are both known. In the present analysis the term β (Equation 4) describes the location of the neutral axis. Examination of the numerical results used to generate Fig. 2 showed that the moment–curvature secant modulus depends strongly on β and only weakly on the exact form of the damage law (see Fig. 3).

If the secant modulus is denoted as

$$\phi = \sigma_{nom}/E\epsilon_{nom}, \tag{7}$$

the curves in Fig. 3 are well approximated by

$$\beta = \tfrac{1}{6}(5\phi + 1) \tag{8}$$

Denoting ϵ as the strain at a distance z below the tensile surface

$$\epsilon = \frac{y_T - z}{\rho} \tag{9}$$

and combining Equations 4, 6, 7 and 9

$$\frac{\epsilon}{\epsilon_{nom}} = 2\left(\frac{6}{5\phi+7} - \frac{z}{H}\right). \tag{10}$$

As discussed below, the value of ϵ at $z = 0.2\,H$ is of particular interest since the stress at that location

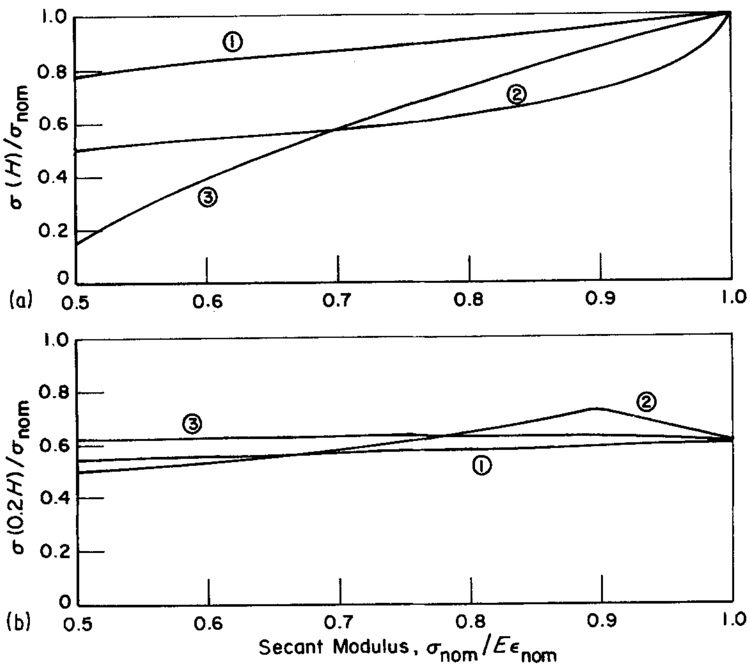

Figure 4 Local stress in a bend bar. (a) tensile surface; (b) 0.2H below tensile surface.

is fairly insensitive to the form of the damage law

$$\epsilon(z = 0.2H) \approx 0.4\epsilon_{nom}\left(\frac{23-5\phi}{5\phi+7}\right). \quad (11)$$

The calculation of local stress requires a different approach from the calculation of local strain. Table I lists the tensile stress distributions associated with the individual damage laws. These distributions can be calculated from Equations 1 to 3 since strain can be calculated from Equation 10. Note that the equations are given in terms of the location of the neutral axis (β), while the calculation below utilizes the experimentally measurable secant modulus, ϕ. In the stress calculation below the individual relations between ϕ and β (Table I) were used instead of the average ϕ/β relation (Equation 8).

The calculation of the stress distribution reveals a striking dependence of outer-fibre stress on damage law (Fig. 4a). The figure shows that the linear-elastic result ($\sigma = \sigma_{nom} = 6M/BH^2$) is valid only if the moment–curvature relationship reflects no damage. Increasing departures of the experimental bend-test record from linearity are accompanied by increasing departures of the outer-fibre stress from σ_{nom}.

A simple reference stress method [8] was used to investigate whether the stress at any interior location is independent of secant modulus. The equations in Table I were used and by trial-and-error the value of $z = 0.2\,H$ was selected with the results given in Fig. 4b. The damage-free linear-elastic value of σ at this location is

$$\sigma(z = 0.2H) = 0.6\,\sigma_{nom} \quad (12)$$

while the differences in different damage laws results in $\sigma = (0.6 \pm 0.1)\sigma_{nom}$.

The above information allows construction of the tensile stress–strain curve. The construction involves measurement of load and displacement for a number of points on the bend-test record. These values are converted into σ_{nom} and ϵ_{nom} using Equations 5 and 6. The associated stresses and strains are then calculated from Equations 11 and 12 to provide the desired tensile curve.

3. Discussion

The approach described in this paper provides a method of estimating the damage law without performing a tensile test. This is an advantage since bend tests are much simpler to perform and much freer from measurement error than tensile tests. However, there are several compensating problems. A principal problem is that the procedure is not completely general. For example, if a material has the same non-linear behaviour in tension and compression, the moment–curvature relationship will also be non-linear but the neutral axis will not migrate. To determine whether a bend bar is in this category one could physically examine the specimen for preferential damage in the tensile region (e.g., cracks, pores). In addition to the above problem, it is possible to construct damage laws in which the neutral axis does not migrate. A simple example is the "perfectly-plastic" damaged material

$$\begin{aligned} \sigma &= \sigma_0; \quad \text{compression} \\ \sigma &= A\sigma_0, A < 1; \quad \text{tension.} \end{aligned} \tag{13}$$

Further research is required to define the constitutive relationships which fall in the latter category.

The final problem with the analysis is that it is not necessarily adequate to define the fracture load of the bend bars. To illustrate this point, consider the "elastic" damage law, Equation 1. As noted above, this law describes the situation where pre-existing cracks open but do not grow. Equation 1 has no terminus within the framework of the model. An additional condition needs to be added: the specimen will fail when the stress intensity on the most dangerous flaw reaches K_{Ic} for the material. An additional complication is that the cracks may begin to extend at some critical stress level due to environmentally assisted growth, thus lowering the specimen compliance and altering the form of the damage law during the test.

This illustration suggests that the other damage models may need analogous modification. More generally it is possible that the mechanism of damage changes at a critical stress or strain and/or a failure criterion needs to be added to the model. In contrast, Krajcinovic [2] has argued that there is only one damage mechanism and that the maximum load in Fig. 2 defines failure. In addition to postulating a simple damage law, this approach does not take the statistical nature of fracture into account. In this view [6], the damage law reflects the defect distribution within the material. However, scatter in fracture strength is not predicted since the damage modulus, D, is tacitly assumed to be an invariant material property. If, however, D is assumed to be statistically distributed, this objection may be overcome.

One possible problem with the analysis is that it does not extract the maximum information from the system, in that it does not provide a quantitative characterization of the stress–strain behaviour in the most highly damaged region of the specimen, namely, the tensile outer fibre. This problem is only of importance if the strain in the outer-fibre region becomes sufficient to activate an additional damage mechanism which is not experienced by the material at the $0.2H$ location. Nevertheless, the amount of information provided from a bend test should be comparable to the amount provided from a tensile test, due to the greater stability of the bend specimen. To illustrate this point, consider the linear-damage law material, Equation 3. By Krajcinovic's [2] analysis tensile test failure would occur at $\epsilon = 0.5D/E$ (Fig. 2), provided the loading system is not too stiff. Failure in a bend test would occur at $\epsilon_{nom} \approx 2/3$ (D/E) (Fig. 2) at this point $\phi \approx 2/3$ (Equation 7). In our analysis the stresses and strains are calculated at a depth $0.2H$, at which point the strain at maximum load from Equation 11 is about $3/4\epsilon_{nom} \approx 0.5D/E$. To the extent that this result is typical, a bend specimen can provide information about the stress–strain curve of a tensile specimen up to the strain at which the tensile specimen would fail. However, it should be reiterated that the maximum load failure criterion is not necessarily correct since it does not take into proper account the statistical nature of fracture of brittle materials.

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Appendix
The method of calculation can be illustrated by deriving the relationships for the "elastic" law (Equation 1):

$$\sigma = E\epsilon, \quad \text{compression} \tag{A1}$$

$$\sigma = AE\epsilon, \quad \text{tension} \tag{A2}$$

Using the notation of Fig. A1 the force in the tension part of the beam F_T is

$$F_T = \int_0^{y_T} \sigma B \, dy \tag{A3}$$

Since y is measured from the neutral axis,

$$y = \frac{y_T \sigma}{AE\epsilon_T} \tag{A4}$$

when ϵ_T and y_T are outer-fibre values.

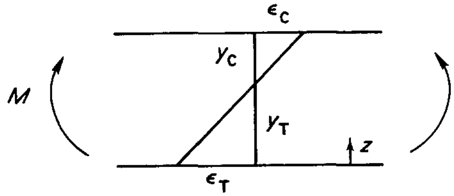

Figure A1 Key to Notation.

Combining Equations A3 and A4

$$F_T = \frac{AE\epsilon_T B y_T}{2}. \qquad \text{(A5)}$$

Similarly, for compression

$$F_C = \frac{E\epsilon_C B y_C}{2}. \qquad \text{(A6)}$$

Since force equilibrium requires $F_T = F_C$

$$A\epsilon_T y_T = \epsilon_C y_C. \qquad \text{(A7)}$$

Letting $\beta \equiv y_C/y_T$, Equation A7 becomes

$$A = \beta^2. \qquad \text{(A8)}$$

The moment balance is

$$\frac{M}{B} = \int_0^{y_C} \sigma y \, dy + \int_0^{y_T} \sigma y \, dy \qquad \text{(A9)}$$

Using the above relationships,

$$\frac{M}{BH^2} = \frac{E\epsilon_T \beta^2}{3(1+\beta)}. \qquad \text{(A10)}$$

Now

$$\epsilon_T = \frac{y_T}{\rho}, \qquad \text{(A11)}$$

where ρ is the radius of curvature of the beam. Recalling the definition of β

$$\epsilon_T = \frac{H}{(1+\beta)\rho}, \qquad \text{(A12)}$$

Equations A10 and A12 can be combined with Equations 5 and 6 of the text to give

$$\frac{\sigma_{nom}}{D} = \frac{4E\epsilon_{nom}}{D}\left(\frac{\beta}{1+\beta}\right)^2. \qquad \text{(A13)}$$

To calculate the stress at any point in the beam, first calculate the strain and express it in terms of z, the distance from the tensile outer fibre

$$\epsilon = \frac{y_T - z}{\rho}. \qquad \text{(A14)}$$

Again recalling the definition of β and using Equations 8 and 10:

$$\epsilon = \frac{2\epsilon_{nom}}{1+\beta}\left[1 - (1+\beta)\frac{z}{H}\right]. \qquad \text{(A15)}$$

Combining Equations A2, A8, A13 and A15:

$$\frac{\sigma}{\sigma_{nom}} = \frac{1}{2}(1+\beta)\left[1 - (1+\beta)\frac{z}{H}\right]. \qquad \text{(A16)}$$

The analogous relationships for the other damage laws were derived by a similar procedure.

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